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The diffusion of a discontinuity of the shear stress at the boundary of a viscoplastic half-plane☆

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Abstract

For the problem of the diffusion of a discontinuity of the shear stress at the boundary of a half-plane, which is a special case of the general problem of the diffusion of a vortex layer, the classes of media and types of assignment of boundary conditions for which self-similar solutions exist are discussed. For a viscoplastic medium in a half-plane the problem reduces to the problem in a layer of time-variable thickness, the solution of which does not possess the property of analyticity. The long-term asymptotic of this problem are investigated. In the case where, at an accessible boundary, it is possible simultaneously to measure both the shear stress and the horizontal velocity, an algorithm is proposed for finding a quantity that is difficult to measure, A namely, the thickness of the zone of viscoplastic flow.

1. The diffusion of a vortex layer in a medium without a yield point

1.1. Self-similarity

Self-similar solutions, which depend on a variable of the type $x/\sqrt{t}$, are classical fundamental solutions of one-dimensional linear and non-linear equations of heat conduction and describe numerous physical effects with a discontinuity at the boundary at the initial instant of time. Self-similarity is a property not only of the given equation but also of the entire initial boundary-value problem specified for it.

The following initial boundary-value problem for a one-dimensional linear parabolic equation is well known (see, for example, Ref. 1)

$$u_t(x,t) = \nu u_{xx}(x,t);\quad \nu>0,\quad u(x,t)\in C^2(D_0),\quad D_0 = \{t>0, x>0\} \tag{1.1}$$

$$u(x,0)\equiv 0,\quad x>0;\quad u(0,t) = U_0 h(t),\quad U_0 = \text{const}, \tag{1.2}$$

where $h(t)$ is Heaviside's function. The initial and boundary conditions (1.2) are obviously mutually inconsistent at the corner point O ($t=0$, $x=0$) of region D_0, since

$$0 = \lim_{x\to 0+} u(x,0) \ne \lim_{t\to 0+} u(0,t) = U_0,$$

which results in there being no limit to the solution when tending towards the point O.

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A precise analytical solution of problem (1.1), (1.2) can be written using a complementary error function and has the self-similar form[1]

$$u(x,t) = U_0 \operatorname{erfc}\frac{x}{2\sqrt{\nu t}}. \tag{1.3}$$

It can be seen that this solution satisfies the necessary requirements of smoothness in the region D_0, and its limit in tending towards the point O actually depends on the trajectory of the approach to this point.

The initial boundary-value problem (1.1), (1.2) quite adequately models many physical effects in one-dimensional structures that have a discontinuity at the boundary at the initial instant of time, such as heat transfer, diffusion, seepage, etc. It also describes the one-dimensional transient shear of a viscous incompressible fluid with kinematic viscosity ν in the half-plane $x_2 > 0$. The velocity vector and the viscous stress tensor under such shear each have one non-zero component: $\upsilon_1(x_2, t) \equiv \upsilon(x, t)$ and $\sigma_{12}(x_2, t) \equiv \sigma(x, t)$. Depending on which quantity, υ or σ, acts as $\upsilon(x, t)$, solution (1.3) is related to the diffusion of a velocity or shear stress discontinuity. It is also traditional for such solutions in hydrodynamics to use the general term "vortex layer diffusion".[2,3] The influence of this vortex layer [or discontinuity caused by instantaneous application of a tangential velocity or shear stress at the boundary of a quiescent half-plane, according to the initial condition (1.2)] extends along the x axis at infinite velocity, which is typical of parabolic problems. Furthermore, the vortex layer diffuses instantaneously when $t > 0$.

The problem of vortex layer diffusion also has an exact analytical solution in the case of a more general specification of boundary condition (1.2), namely

$$u(0,t) = U(t),$$

where $U(t)$ is a piecewise-continuous function, equal to zero when $t < 0$ with discontinuities of the first kind at points $t_1, t_2, \ldots, t_i, \ldots$ $(i \geq 0)$. This solution is written using Stieltjes integral

$$u(x,t) = \int_0^t \operatorname{erfc}\frac{x}{2\sqrt{\nu(t-\tau)}}dU(\tau). \tag{1.4}$$

Here, $dU(t) = (\dot{U}(t) + \sum_i [U]_i \delta(t - t_i))dt$, where $[U]_i$ is the jump of the function U at the point of discontinuity t_i.

Eq. (1.1), both when $u = \upsilon(x, t)$ and $u = \sigma(x, t)$, is the obvious consequence of a system consisting of the equation of motion

$$\sigma_x = \rho\upsilon_t, \quad (t,x) \in D_0 \tag{1.5}$$

and the constitutive relation for a viscous Newtonian fluid ($\mu = \rho\nu$)

$$\sigma = \mu\upsilon_x \quad (t,x) \in D_0. \tag{1.6}$$

In the case of a viscous Newtonian medium the shear flow, as before, occurs in the region D_0, but the material is characterized by a non-linear rheological law

$$\sigma = F(\upsilon_x), \quad F(0) = 0. \tag{1.7}$$

From system (1.5), (1.7) we have the non-linear parabolic equations

$$\rho\upsilon_t = F'(\upsilon_x)\upsilon_{xx}, \quad \rho(F^{-1})'(\sigma)\sigma_t = \sigma_{xx} \quad (t,x) \in D_0. \tag{1.8}$$

After introducing the self-similar variable $\eta = x/(2\sqrt{\nu_0 t})$, where ν_0 is the intrinsic kinematic viscosity in non-linear model (1.7), Eqs. (1.8) reduce to ordinary differential equations in the functions $\psi(\eta) = \upsilon_x(x, t)$ and $\varphi(\eta) = \sigma(x, t)$, $0 < \eta < \infty$:

$$\frac{d}{d\eta}\left(F'(\psi)\frac{d\psi}{d\eta}\right) + 2\rho\nu_0\eta\frac{d\psi}{d\eta} = 0, \quad \frac{d^2\varphi}{d\eta^2} + 2\rho\nu_0\eta(F^{-1})'(\varphi)\frac{d\varphi}{d\eta} = 0. \tag{1.9}$$

By specifying the boundary conditions in the form $\psi(0) = V_0'$, $\psi(\infty) = 0$ or $\varphi(0) = S_0$, $\varphi(\infty) = 0$, following from conditions (1.2), in addition to Eq. (1.8), we can make self-similar the problems of vortex layer diffusion in a non-linearly viscous medium with an arbitrary rheological law F both in terms of strain rates and in terms of stresses.

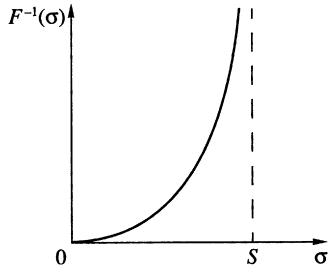

Fig. 1.

We will give an example of an exact analytical solution of the second equation of system (1.9). We will select a two-constant non-linearly viscous fluid with a function F such that the constitutive relation, inverse to relation (1.6), has the form (Fig. 1, the continuous curve)

$$\upsilon_x = F^{-1}(\sigma) = -\frac{S}{\mu}\left[\ln\left(1-\frac{\sigma}{S}\right)+\frac{\sigma}{S}+\frac{\sigma^2}{2S^2}\right], \quad 0 \le \sigma < S. \tag{1.10}$$

The shear stresses in model (1.10) cannot exceed the material constant S, and as $\sigma \to S-0$ the strain rate tends to infinity. In a certain sense, the behaviour described by rheological model (1.10) is similar to the behaviour of a rigid perfectly plastic material with limit stress S (Fig. 1, the dashed line).

Substituting expression (1.9) into the second equation of system (1.8) and considering that the intrinsic kinematic viscosity ν_0 in the given case is equal to μ/ρ, we obtain an ordinary differential equation in $\varphi(\eta)$

$$\frac{d^2\varphi}{d\eta^2}+\frac{2\eta\varphi^2}{S(S-\varphi)}\frac{d\varphi}{d\eta} = 0. \tag{1.11}$$

Suppose the prescribed shear stress S_0 at the boundary of the half-plane is identical with S. Then, the exact solution of Eq. (1.11) with the boundary conditions

$$\varphi(0) = S, \quad \varphi(\infty) = 0$$

in the entire half-plane is as follows:

$$\varphi = \frac{S}{1+\eta}.$$

This means that

$$\sigma(x,t) = S\left(1+\frac{x}{2\sqrt{\nu_0 t}}\right)^{-1}. \tag{1.12}$$

2. The presence of a yield point and the absence of self-similarity

Now suppose the material has a yield point for shear, τ_s, and exhibits viscoplastic properties. We will include in the basis of the scales for transition to dimensionless variables the density ρ, the dynamic viscosity μ and the quantity τ_s, so that we consider all subsequent relations in Sections 2 and 3 to be written in dimensionless form.

As is well known, viscoplastic flow occurs where and when the quadratic invariant of the stress deviator, equal in the given problem to $|\sigma|$, is greater than τ_s. The remaining part of the material belongs to the rigid zone where the quadratic invariant of the strain rates $\sqrt{2v_{ij}v_{ij}}$, which in the given problem is equal to $|\upsilon_x|$, is zero. The interface of the flow zone and the rigid zone in which $|\sigma| = 1$ is subject to determination on a par with other required quantities and is often of main interest in problems of mass and heat transfer.[4,5]

We will investigate the diffusion of the shear stress discontinuity, which is specified by the conditions

$$\sigma(x, 0^+) = 0, \quad x > 0 \quad \sigma(0,t) = S_0 h(t), \quad S_0 = \text{const}. \tag{2.1}$$

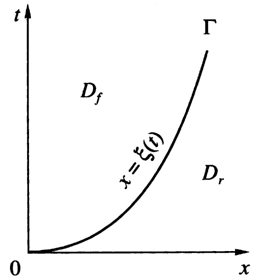

Fig. 2.

Here (see Fig. 2)

$$D_0 = D_f \cup \Gamma \cup D_r, \quad D_f = \{t>0, 0<x<\xi(t)\}$$

$$\Gamma = \{t>0, x=\xi(t)\}, \quad D_r = \{t>0, x>\xi(t)\},$$

where D_f is the band of viscoplastic shear and Γ is the boundary separating the rigid half-plane D_r from D_f. The following condition is imposed on the monotonically increasing continuous function $\xi(t)$ ($\xi(0^+)=0$)

$$|\sigma(\xi(t), t)| = 1. \tag{2.2}$$

Thus, it is required to find the functions

$$\upsilon \in C^1(D_0), \quad \upsilon \in C^2(D_f, D_r), \quad \sigma \in C^1(D_0), \quad \sigma \in C^2(D_f, D_r),$$

satisfying the systems

$$\sigma_x = \upsilon_t, \quad |\sigma| = 1+|\upsilon_x| \quad (t,x) \in D_f \tag{2.3}$$

$$\sigma_x = \upsilon_t, \quad \upsilon_x = 0 \quad (t,x) \in D_r, \tag{2.4}$$

and also condition (2.2). The second equality of (2.3) is the constitutive relation for a viscoplastic medium with linear viscosity, or a Shvedov–Bingham medium.

The initial boundary-value problem (2.1)–(2.4) can be classified as a parabolic problem in a region with an unknown moving boundary.[6,7] It can be reformulated in new independent variables $y=x/\xi(t)$ and $\tilde{t} = t$ (the tilde over the t is omitted below)

$$\sigma_y = \xi\upsilon_t - \dot{\xi}y\upsilon_y, \quad |\upsilon_y| = \xi(|\sigma|-1) \quad (t,y) \in D'_f \tag{2.5}$$

$$\sigma_y = \xi\upsilon_t, \quad \upsilon_y = 0 \quad (t,y) \in D'_r \tag{2.6}$$

$$|\sigma(1,t)| = 1, \quad \sigma(0,t) = S_0 h(t), \quad \sigma(y, 0^+) = 0 \tag{2.7}$$

$$D'_f = \{t>0, 0<y<1\}, \quad D'_r = \{t>0, y>1\}.$$

Problem (2.5)–(2.7) is formulated in regions with fixed boundaries, and the unknown function ξ has passed from the boundary conditions to the equations.

To fix our ideas, suppose $S_0>0$, Then $\sigma>0$ and $\upsilon_y>0$ in the region D'_f. Here, problem (2.5)–(2.7) can be formulated in stresses as follows:

$$\sigma_{yy} + \xi\dot{\xi}y\sigma_y = \xi^2\sigma_t \quad (t,y) \in D'_f \tag{2.8}$$

$$\sigma_{yy} = 0 \quad (t,y) \in D'_r \tag{2.9}$$

$$\sigma(1,t) = 1, \quad \sigma(0,t) = S_0, \quad \sigma(y,0^+) = 0. \tag{2.10}$$

In an attempt to find a solution of the initial boundary-value problem (2.8)–(2.10) in self-similar form, which is equivalent to the case where the function σ depends only on the variable y, from Eq. (2.8) we have the ordinary differential equation

$$\frac{d^2\sigma}{dy^2} + by\frac{d\sigma}{dy} = 0, \quad 0<y<1, \tag{2.11}$$

where $\dot{\xi}\xi = b = \text{const}$, so that the dependence of the thickness of the flow zone on t is immediately determined, apart from the constant, as follows: $\xi = \sqrt{2bt}$. Eq. (2.11) with boundary conditions (2.10) [$\sigma(0)=S_0$, $\sigma(1)=1$] has the solution

$$\sigma = S_0\left(1 - \frac{C\sqrt{\pi}}{2}\operatorname{erf}\left(y\sqrt{\frac{b}{2}}\right)\right). \tag{2.12}$$

The constants $C>0$ and b are related by the condition that the right-hand side of relation (2.12) is equal to unity when $y=1$.

From this condition it can be seen that, if $0 \le S_0 \le 1$, then C cannot be selected, i.e. there is no flow zone in this case, and, when $t>0$, the entire half-plane is covered by a rigid zone. In fact, there are insufficient shear forces $|S_0| \le 1$ at the boundary to overcome the yield point. Thus, the condition of feasibility of viscoplastic flow has the form $|S_0|>1$.

Basing on the stress (2.12), from system (2.5) we obtain the velocity in the flow zone $0<y<1$:

$$\upsilon(y,t) = \left[(S_0-1)y - \frac{S_0\sqrt{\pi}Cy}{2}\operatorname{erf}\left(y\sqrt{\frac{b}{2}}\right) - \frac{S_0C}{\sqrt{2b}}\exp\left(-\frac{by^2}{2}\right)\right]\xi(t), \tag{2.13}$$

and also the velocity of the boundary of the half-plane $y=0$ and the velocity of the interface $y=1$:

$$\upsilon(0,t) = -\frac{S_0C}{\sqrt{2b}}\xi(t) = -S_0C\sqrt{t} \tag{2.14}$$

$$\upsilon_*(t) \equiv \upsilon(1,t) = -\frac{S_0C}{\sqrt{2b}}\exp\left(-\frac{b}{2}\right)\xi(t) = -S_0C\sqrt{t}\exp\left(-\frac{b}{2}\right). \tag{2.15}$$

Consequently, the rigid half-plane should perform translational motion at a velocity υ_* (2.15) and acceleration

$$\dot{\upsilon}_*(t) = -\frac{S_0C}{\sqrt{2b}}\exp\left(-\frac{b}{2}\right)\dot{\xi}(t) = -\frac{S_0C}{2\sqrt{t}}\exp\left(-\frac{b}{2}\right). \tag{2.16}$$

We will return to the stress and velocity distributions in the rigid zone, described by Eq. (2.9), which, by virtue of the continuity of σ when $y>0$, and also self-similarity, has the integral

$$\sigma(y) = 1 - k(y-1), \quad y>1.$$

However, by definition, at any point of the rigid zone $|\sigma| \le 1$, and therefore $k=0$ and $\sigma \equiv 1$, and from the equation of motion (2.6) it immediately follows that $\upsilon_t \equiv \dot{\upsilon}_*(t) \equiv 0$. As can be seen, this contradicts the derived dependence (2.16) of the acceleration on t, i.e. system (2.8)–(2.10) does not have a self-similar solution $\sigma(y)$.

The absence of self-similarity in the problem of vortex layer diffusion in a Shvedov–Bingham medium possessing a yield point, unlike similar problems for linear and non-linear viscous fluids, is due to the fact that flow occurs in the expanding band D'_f with an unknown boundary (Fig. 2), rather than in the region D_0.

One of the first problems of viscoplasticity theory, in which account was taken of the motion of the boundary of the rigid core, was the problem of the impact of a viscoplastic rod against a solid obstacle.[8] Plane-parallel and axisymmetric shear of a Shvedov–Bingham material in regions with moving boundaries has been investigated in Refs. 9,10. There have been reviews of the plane and three-dimensional transient deformation of similar media in non-canonical regions, and also on problems of the stability of this deformation.[4,5,11–13]

3. The problem in a region with an unknown boundary and non-analyticity of its solution

3.1. Long-term asymptotics

A consequence of the required smoothness of the functions $\sigma(y, t)$ and $\upsilon(y, t)$ on changing from region D'_f to D'_r is the equality $\upsilon_t(1, t) = 0$. Combined with the first equation of system (2.5) and the condition $\upsilon_y(1, t) = 0$, this means that $\sigma_y(1, t) = 0$. The latter condition is additional to the formulation of problem (2.8)–(2.10). In region D'_r, as shown above, the solution is trivial ($\sigma \equiv 1$, $\upsilon_t \equiv 0$), and in region D'_f we have the following problem for the two unknown functions $\sigma(y, t)$ and $\xi(t)$

$$\begin{aligned} &\sigma_{yy} + \xi\dot{\xi}y\sigma_y = \xi^2\sigma_t \\ &\sigma(0, t) = S_0, \quad \sigma(1, t) = 1, \quad \sigma_y(1, t) = 0, \quad \xi(0) = 0. \end{aligned} \tag{3.1}$$

The initial condition for σ is omitted by virtue of the fact that, at $t = 0$, the region D'_f is absent and the replacement $(x, t) \to (y, t)$ becomes degenerate.

Recently, a problem similar to (3.1) was investigated,[14] and the conditions affecting the existence of a viscoplastic flow zone, and also the finiteness of the characteristics of short-term flow, were discussed.

Problem (3.1), which can be formulated in variables (x, t) in the form

$$\begin{aligned} &\sigma_{xx} = \sigma_t, \quad 0 < x < \xi(t) \\ &\sigma(0, t) = S_0, \quad \sigma(\xi(t), t) = 1, \quad \sigma_x(\xi(t), t) = 0, \quad \xi(0) = 0 \end{aligned} \tag{3.2}$$

is one of the versions of the well-known Stefan problem.[1,15] It is easy to show that all partial derivatives with respect to y of the solution $\sigma(y, t)$ of problem (3.1) at the point $y = 1$, just like all partial derivatives with respect to x of the solution $\sigma(x, t)$ of problem (3.2) on the curve $x = \xi(t)$, are zero for any t. Nonetheless, in the viscoplastic flow zone, these functions are not identically equal to their values at $y = 1$ [or at $x = \xi(t)$], i.e. to unity. This indicates the non-analyticity of the solutions $\sigma(y, t)$ and $\sigma(x, t)$ of problems (3.1) and (3.2).

We will investigate the possibility of specifying on the boundary of the half-plane, instead of the first condition in problem (3.2), of the following regime of the shear stress

$$S(t) = S_0 h(t) + S_1(t),$$

where $S_1(t)$ is a continuous function, if only in a small time interval $0 < t < t_\varepsilon$, satisfying the requirements $S_1(0) = 0$ and $\dot{S}_1(t) \geq 0$, so that the solution of Stefan's problem has a form similar to (1.4):

$$\sigma(x, t) = \int_0^t \operatorname{erfc}\frac{x}{2\sqrt{t-\tau}}dS(\tau). \tag{3.3}$$

Here, two other boundary conditions for σ lead to a system of integral equations in the two functions $\xi(t)$ and $S_1(t)$

$$\begin{aligned} &S_0 \operatorname{erfc}\frac{\xi(t)}{2\sqrt{t}} + \int_0^t \operatorname{erfc}\frac{\xi(t)}{2\sqrt{t-\tau}}dS_1(\tau) = 1 \\ &\frac{S_0}{\sqrt{t}}\exp\left(-\frac{\xi^2(t)}{4t}\right) + \int_0^t \exp\left(-\frac{\xi^2(t)}{4(t-\tau)}\right)\frac{dS_1(\tau)}{\sqrt{t-\tau}} = 0. \end{aligned} \tag{3.4}$$

Both terms on the left-hand side of the last equation of system (3.4) for as small a $t < t_\varepsilon$ as desired are strictly positive, and therefore their sum cannot be zero. Consequently, the presence of an additional boundary condition $\sigma_x(\xi(t), t) = 0$ makes system (3.4) incompatible and leads to a negative answer to the question of the admissibility of solution (3.3).

Let us outline ways of constructing asymptotic solutions of problem (3.1) when $t \to \infty$. We will seek the solution of this problem in the form

$$\sigma(y, t) = 1 + (S_0 - 1)Y_1(y)\exp(-T(t)Y(y)) \tag{3.5}$$

with three unknown functions $T(t)$, $Y(y)$ and $Y_1(y)$. The last two should satisfy the conditions

$$Y(0) = 0, \quad Y(1) = \infty, \quad Y_1(0) = 1, \quad Y_1(1) = 0. \tag{3.6}$$

The long-term asymptotic behaviour of the solution corresponds to $T(t) \sim t^{-q}$ and $\xi(t) \sim \sqrt{t}$, where the unknown exponent q is positive. Along with q, it is necessary to find functions Y and Y_1 from the system

$$Y_1'' + yY_1'/2 = 0, \quad Y_1Y'' + (2Y_1' + yY_1/2)Y' + qY_1Y = 0 \tag{3.7}$$

obtained after substituting representation (3.5) into the first equation of system (3.1) and selecting the main terms of the asymptotic forms. The function $Y_1(y)$ is found immediately from the first equation of system (3.7) with boundary conditions (3.6):

$$Y_1(y) = 1 - \frac{\operatorname{erf}(y/2)}{\operatorname{erf}(1/2)}.$$

Then, the exponent q is an eigenvalue, while the function $Y(y)$ is an eigenfunction of the linear problem for the eigenvalues

$$Y'' + \left(\frac{y}{2} - \frac{2\exp(-y^2/4)}{\sqrt{\pi}(\operatorname{erf}(1/2) - \operatorname{erf}(y/2))}\right)Y' + qY = 0, \quad Y(0) = 0, \quad Y(1) = \infty.$$

4. Finding the function $\xi(t)$ from measurements on the accessible boundary $y = 0$

We will return to dimensional variables and everywhere below we will retain after them the notation of the corresponding dimensionless variables. After applying at the initial instant of time on the boundary of the half-plane $y = 0$ the shear stress

$$S_0 = \alpha_1\tau_s, \quad \alpha_1 > 1 \tag{4.1}$$

it is possible to measure the velocity of the given boundary. According to relation (2.14), the modulus of this velocity increases with time in proportion to $\sqrt{t}$ and, consequently, has the form

$$|v(0, t)| = \alpha_2\tau_s\sqrt{\nu t}/\mu, \tag{4.2}$$

where α_2 is an experimentally obtained proportionality factor. Furthermore, expression (2.14) itself will be rewritten as follows:

$$|v(0, t)| = S_0C\sqrt{\nu t}/\mu. \tag{4.3}$$

Now comparing (4.2) and (4.3) and taking into account Eq. (4.1), we arrive at an expression for the dimensionless constant C:

$$C = \alpha_2/\alpha_1. \tag{4.4}$$

Since the coefficients α_1 and α_2 are known from measurements on the accessible boundary $y = 0$, the value of C is known. We substitute this value into the relationship of C and b, which, in dimensional variables, will take the form

$$S_0\left(1 - \frac{C\sqrt{\pi}}{2}\operatorname{erf}\sqrt{\frac{b}{2\nu}}\right) = \tau_s \tag{4.5}$$

and we arrive at an algebraic equation in the parameter b

$$\frac{\sqrt{\pi}}{2}\operatorname{erf}\sqrt{\frac{b}{2\nu}} = \frac{\alpha_1 - 1}{\alpha_2}. \tag{4.6}$$

Knowing the value of parameter b, we find the actual thickness of the flow zone

$$\xi(t) = \sqrt{2bt}. \tag{4.7}$$

We will give a numerical example in which the material constants ρ, μ and τ_s are taken to be characteristic of ice structures and are often used in glaciology calculations:[16] $\rho \approx 10^3$ kg/m^3, $\mu \approx 10^{11}$ kg/m s (for natural ice, with a force perpendicular to the main crystallographic axis) and $\tau_s \approx 10^3$ kg/m s^2. Suppose that, after the instantaneous application at the upper boundary of a shear load of 2×10^3 kg/m s^2 ($\alpha_1 = 2$), the velocity of this boundary after 1 s amounted to 10^{-3} m/s. We will estimate from these data the value of the parameter b.

According to relation (4.2), $\alpha_2 \approx 10$, and from formula (4.6) we find $b \approx 2 \times 10^6$ m^2/s. Using equality (4.7), we conclude that, for example, 1 s after the onset of motion, the thickness of the viscoplastic flow zone will be about 2 km.

Similar problems are encountered in the soil mechanics when modelling the motion of landslides.[17]

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